Conformal Field Theory and Gravity

Solutions to Problem Set 9

Fall 2024

1. Basics of 2D CFT

(a) By symmetry, we can assume that all indices are ordered, so there are at most $\ell + 1$ components $M_{11\ldots 1}$, $M_{11\ldots 12}$, …, $M_{2\ldots 2}$. Tracelessness implies that

$$\delta_{\mu\nu} M_{\mu\nu\alpha\ldots\beta} = M_{11\alpha\ldots\beta} + M_{22\alpha\ldots\beta} = 0 \tag{1}$$

hence we can eliminate any pair of (11) indices in favor of a pair of (22) indices. Thus any tensor with spin $\ell \geq 2$ has 2 independent components: $M_{122\ldots 2}$ and $M_{2\ldots 2}$ (the exception is $\ell = 0$, which has a single component).

We still need to show that $M_{z\ldots z} := M$ and $M_{\bar{z}\ldots\bar{z}} := \bar{M}$ are independent. In fact, we can simply show that all other components vanish. This is again due to tracelessness. In the z, $\bar{z}$ coordinates, the flat-space metric reads $ds^2 = dz d\bar{z}$, so the components of the metric are

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0 \tag{2}$$

and the inverse metric satisfies $g^{zz} = g^{\bar{z}\bar{z}} = 0$, $g^{z\bar{z}} = g^{\bar{z}z} = 2$. Thus

$$0 = g^{\mu\nu} M_{\mu\nu\alpha\ldots\beta} = 4 M_{z\bar{z}\alpha\ldots\beta} \tag{3}$$

for any indices $\alpha, \ldots, \beta$. In conclusion, the only non-zero components of any traceless symmetric tensor are M and $\bar{M}$. Finally, using the Jacobian, we find that

$$M_{12\ldots 2} = i^{\ell-1}\left(M + (-1)^{\ell-1}\bar{M}\right) \quad \text{and} \quad M_{2\ldots 2} = i^{\ell}\left(M + (-1)^{\ell}\bar{M}\right). \tag{4}$$

Conservation of a tensor means that

$$0 = g^{\mu\nu}\partial_\mu M_{\nu\alpha\ldots\beta} = 2\left(\partial\bar{M}_{\alpha\ldots\beta} + \bar{\partial} M_{\alpha\ldots\beta}\right) \tag{5}$$

using the shorthand notation $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$. In particular, by setting all coordinates either equal to z or to $\bar{z}$, we find that

$$\bar{\partial} M = 0, \quad \partial\bar{M} = 0. \tag{6}$$

In other words, the component M depends only on z, and $\bar{M}$ only depends on $\bar{z}$.

Under a finite rotation R, a tensor transforms as

$$T^{\mu\ldots\nu}(x) \mapsto R^{\mu'}_{\mu} \ldots R^{\nu'}_{\nu} T^{\mu'\ldots\nu'}(x'). \tag{7}$$

In the z, $\bar{z}$ coordinates, a rotation by an angle θ can be represented as

$$R^{\mu}_{\nu} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{8}$$

Thus

$$M(z,\bar{z}) \mapsto e^{i\ell\theta} M(z',\bar{z}'), \quad \bar{M}(z,\bar{z}) \mapsto e^{-i\ell\theta}\bar{M}(z',\bar{z}'), \tag{9}$$

with $z' = e^{i\theta} z$, $\bar{z}' = e^{-i\theta}\bar{z}$.

Finally, in the z, $\bar{z}$ coordinates, parity ($y \mapsto -y$) acts as

$$P^{\mu}_{\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

so that (up to an intrinsic parity $\eta_M = \pm 1$),

$$M(z,\bar{z}) \mapsto \bar{M}(\bar{z},z), \quad \bar{M}(z,\bar{z}) \mapsto M(\bar{z},z). \tag{11}$$

(b) We can start from the known 2-pt function

$$\langle J_\mu(x) J_\nu(y)\rangle = k_J \frac{I_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} \tag{12}$$

for some constant $k_J > 0$, where for conserved currents, obviously $\Delta = d-1 = 1$. We can get the component correlators using

$$J = (\partial x^\mu) J_\mu = \frac{1}{2}(J_1 - iJ_2), \quad \bar{J} = (\bar{\partial} x^\mu) J_\mu = \frac{1}{2}(J_1 + iJ_2). \tag{13}$$

Then we find that

$$\langle J(z,\bar{z}) J(w,\bar{w})\rangle = -\frac{k_J}{2}\frac{1}{(z-w)^2}. \tag{14}$$

Setting $\Delta = 1$, we find in particular that

$$\langle J(z,\bar{z}) J(w,\bar{w})\rangle = -\frac{k_J}{2(z-w)^2} \tag{15}$$

so we confirm that the correlator only depends on the holomorphic coordinates z, w. Likewise

$$\langle \bar{J}(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = -\frac{k_J}{2(\bar{z}-\bar{w})^2}, \quad \langle J(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = 0. \tag{16}$$

Parity is automatically preserved in this way. Conversely, without parity invariance we could write the same 2-pt functions with different constants k_J in the $\langle JJ\rangle$ and $\langle \bar{J}\bar{J}\rangle$ correlators, and such correlators would be conformally invariant.

Likewise,

$$\langle \bar{J}(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = -\frac{k_J}{2}\frac{1}{(\bar{z}-\bar{w})^2}, \quad \langle J(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = 0. \tag{17}$$

For the stress-energy tensor, we obtain

$$\langle T(z) T(w)\rangle = \frac{c/2}{(z-w)^4}, \quad \langle T(z)\bar{T}(w)\rangle = 0, \quad \langle \bar{T}(z)\bar{T}(w)\rangle = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4}. \tag{18}$$

(c) It is trivial to show that the modes with labels $m, n \in \{-1, 0, 1\}$ form a subalgebra. In particular, their commutator only gives modes in $\{-1, 0, 1\}$. It is a standard fact from complex analysis that the only conformal transformations of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0. \tag{19}$$

The group of these transformations is called the Möbius or global conformal group, denoted by $\mathrm{PSL}(2, \mathbb{C})$. It is generated by translations, rotations, dilations, and inversions:

$$f_{\mathrm{tr}}(z) = z + a, \quad f_{\mathrm{rot}}(z) = e^{i\theta} z, \quad f_{\mathrm{dil}}(z) = cz, \quad f_{\mathrm{inv}}(z) = \frac{1}{z}. \tag{20}$$

The global conformal group enlarges the group of rigid transformations of $\mathbb{C}$ (that is, translations and rotations) by adding scale transformations and mappings that turn the complex plane "inside out." An interesting fact is that $\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{SO}(3, 1)$.

The global Virasoro generators have physical interpretations:

- $L_0 + \bar{L}_0$ generates dilatations and is the Hamiltonian.
- $L_0 - \bar{L}_0$ generates rotations and is the angular momentum.
- L_{-1} and $\bar{L}_{-1}$ generate translations.
- L_1 and $\bar{L}_1$ generate special conformal transformations.

(d) Since the generator of translation is the energy-momentum tensor, we have

$$H = L_0 + \bar{L}_0, \tag{21}$$

which means that h and $\bar{h}$ are the energies of the states.

Let $|\psi\rangle \in \mathcal{H}_{\mathrm{CFT}}$ be an eigenvector of L_0 with weight h. Using the Virasoro algebra, we find that the state $L_n|\psi\rangle$ is also an eigenvector of L_0 with eigenvalue shifted by n:

$$L_0(L_n|\psi\rangle) = ([L_0, L_n] + L_n L_0)|\psi\rangle = (h - n)L_n|\psi\rangle. \tag{22}$$

Since L_n lowers the energy of a state by n, and the Hamiltonian is bounded from below, there must exist an L_0 eigenstate $|\psi\rangle$ that is annihilated by L_n for all $n \geq 0$. Any state $|\psi\rangle \in \mathcal{H}_{\mathrm{CFT}}$ for which

$$L_0|\psi\rangle = h|\psi\rangle, \quad L_n|\psi\rangle = 0 \quad \text{for all } n > 0 \tag{23}$$

is called a primary or highest weight state.

(e) At level 1, there is a single state $L_{-1}|O\rangle$ with norm

$$\langle O|L_1 L_{-1}|O\rangle = 2\langle O|L_0|O\rangle = 2h\langle O|O\rangle = 2h, \tag{24}$$

where we normalized the state to have norm 1. This implies that for unitarity, we need $h > 0$.

At level 2, there are two possible states:

$$|\psi_{1,1}\rangle = L_{-1}^2|O\rangle, \quad |\psi_2\rangle = L_{-2}|O\rangle. \tag{25}$$

The 2×2 Gram matrix at level 2 is

$$M^{(2)}(c,h) = \begin{pmatrix} \langle\psi_{1,1}|\psi_{1,1}\rangle & \langle\psi_{1,1}|\psi_2\rangle \\ \langle\psi_2|\psi_{1,1}\rangle & \langle\psi_2|\psi_2\rangle \end{pmatrix} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & \frac{c}{2}+4h \end{pmatrix}. \tag{26}$$

At level 3, we have

$$|\psi_1\rangle = L_{-1}|O\rangle, \tag{27}$$

with the 3×3 Gram matrix

$$M^{(3)}(c,h) = \begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_{1,1}\rangle & \langle\psi_1|\psi_2\rangle \\ \langle\psi_{1,1}|\psi_1\rangle & \langle\psi_{1,1}|\psi_{1,1}\rangle & \langle\psi_{1,1}|\psi_2\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_{1,1}\rangle & \langle\psi_2|\psi_2\rangle \end{pmatrix} = \begin{pmatrix} 2h & 0 & 0 \\ 0 & 4h(2h+1) & 6h \\ 0 & 6h & \frac{1}{2}(c+8h) \end{pmatrix}. \tag{28}$$

This matrix should be positive definite. One can solve for its eigenvalues. You can read about it in Di Francesco page 207. The trace is

$$\mathrm{Tr}\left(M^{(3)}(c,h)\right) = \frac{c}{2} + 2h(5+4h) \to \frac{c}{2} \text{ when } h = 0 \tag{29}$$

and the determinant

$$\det\left(M^{(3)}(c,h)\right) = 4h(c + 2h(c+8h-5)) \to 0 \text{ when } h = 0. \tag{30}$$

The eigenvalues, with $h = 0$, are 0, 0, and $c/2$, such that we get $c > 0$ again. Null states are states for which norms are zero.

2. OPE and free scalars

(a) We can prove this result using the sampling property of the delta function. Let R be a closed domain in the complex plane. Then the divergence theorem in complex coordinates says that

$$\int_R d^2z(\partial v_z + \overline{\partial} v_{\overline{z}}) = i \oint_{\partial R} (v_z d\overline{z} - v_{\overline{z}} dz), \tag{31}$$

where v_α is a vector field, and the contour ∂R is traversed anticlockwise.

Now let $f(z)$ be a holomorphic test function and suppose that the region R encloses the origin.

$$\int_R d^2z\, \partial\overline{\partial} \ln|z|^2 f(z) = \int_R d^2z\, \overline{\partial}\left(\frac{1}{z} f(z)\right) = -i \oint_{\partial R} \frac{dz}{z} f(z) = 2\pi f(0), \tag{32}$$

where we use the residue theorem in the final line.

Similarly, for an antiholomorphic test function $f(\overline{z})$,

$$\int_R d^2z\, \partial\overline{\partial} \ln|z|^2 f(\overline{z}) = i \oint_{\partial R} \frac{d\overline{z}}{\overline{z}} f(\overline{z}) = 2\pi f(0). \tag{33}$$

This means that

$$\partial\overline{\partial} \ln|z|^2 = 2\pi\delta(z,\overline{z}), \tag{34}$$

by the definition of the delta function.

(b) The free field $X^\mu(z,\bar{z})$ satisfies the OPE

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \tag{35}$$

Taking derivatives, this implies

$$\partial X^\mu(z)\partial X^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{(z-w)^2}. \tag{36}$$

We can invert the mode expansion $\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu z^{-n-1}$ via

$$i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}z^n\partial X^\mu = \oint\frac{dz}{2\pi i}\sum_m \alpha_m^\mu z^{n-m-1} = \sum_m \alpha_m^\mu \delta_{m,n} = \alpha_n^\mu. \tag{37}$$

Then, we can use the OPE (which has implicit radial ordering) to find

$$[\alpha_m^\mu, \alpha_n^\nu] = -\frac{2}{\alpha'}\oint_{w=0}\frac{dw}{2\pi i}w^n\oint_{z=w}\frac{dz}{2\pi i}z^m R(\partial X^\mu(z)\partial X^\nu(w)), \tag{38}$$

$$= \oint_{w=0}\frac{dw}{2\pi i}w^n\oint_{z=w}\frac{dz}{2\pi i}\eta^{\mu\nu}\frac{z^m}{(z-w)^2}, \tag{39}$$

$$= \oint_{w=0}\frac{dw}{2\pi i}w^n\oint_{z=w}\frac{dz}{2\pi i}\eta^{\mu\nu}\frac{mz^{m-1}}{z-w}, \tag{40}$$

$$= \oint_{w=0}\frac{dw}{2\pi i}mw^{m+n-1}\eta^{\mu\nu}, \tag{41}$$

$$= m\eta^{\mu\nu}\delta_{m+n,0}, \tag{42}$$

where we integrated by parts in the third line.

(c) The holomorphic stress tensor is

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : (z),$$

and we will need the OPE

$$\partial X^\mu(z)X^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{z-w}.$$

Then we compute

$$T(z)X^\mu(w) = -\frac{1}{\alpha'} : \partial X^\nu \partial X_\nu(z) : X^\mu(w) \tag{43}$$

$$= -\frac{2}{\alpha'} : \partial X^\nu[\partial X_\nu(z) : X^\mu(w)] + \dots \tag{44}$$

$$= \frac{\partial X^\mu(z)}{z-w} + \dots = \frac{\partial X^\mu(w)}{z-w} + \dots \tag{45}$$

where the [...] denotes a resolution of the OPE.
To find the OPE with $\partial^n X^\mu(w)$ we simply differentiate n times with respect to w. By the Leibniz rule,

$$T(z)\partial^n X^\mu(w) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\partial^{k+1}X^\mu(w)\partial_w^{n-k}\left(\frac{1}{z-w}\right) + \dots \tag{46}$$

$$= \sum_{k=0}^{n} \frac{n!}{k!}\frac{\partial^{k+1}X^\mu(w)}{(z-w)^{n-k+1}} + \dots \tag{47}$$

The second-order pole is at $k = n-1$ and has coefficient $n\partial^n X^\mu(w)$, implying the conformal weight $h = n$. Since $\partial^n X^\mu(w)$ is holomorphic, it is also clear that $\bar{h} = 0$. However, when $n > 1$, the operator is not primary since it contains poles of order greater than 2.

3. Weyl transformations are anomalous in $d = 2$

(a) The Ricci scalar of the metric $ds^2 = e^{2\sigma(z,\bar{z})}dzd\bar{z}$ is

$$R = -8e^{-2\sigma}\partial_z\partial_{\bar{z}}\sigma\,. \tag{48}$$

Thus the Weyl anomaly condition gives

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = 2g^{z\bar{z}}\langle T_{z\bar{z}}\rangle = \frac{c}{24\pi}\left(-8e^{-2\sigma}\partial_z\partial_{\bar{z}}\sigma\right)\,, \tag{49}$$

which implies

$$\langle T_{z\bar{z}}\rangle = -\frac{c}{12\pi}\partial_z\partial_{\bar{z}}\sigma\,. \tag{50}$$

The conservation can be written as

$$g^{\mu\rho}\nabla_\rho\langle T_{\mu\nu}\rangle = 0 \tag{51}$$

To obtain an equation for $\langle T_{zz}\rangle$, choose $\nu = z$. This gives

$$g^{\bar{z}z}\nabla_z\langle T_{\bar{z}z}\rangle + g^{z\bar{z}}\nabla_{\bar{z}}\langle T_{zz}\rangle = 0 \tag{52}$$

The only non-vanishing Christofell symbols are

$$\Gamma^z_{zz} = 2\partial_z\sigma \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}}\sigma\,. \tag{53}$$

The conservation thus gives

$$\partial_{\bar{z}}\langle T_{zz}\rangle = -\partial_z\langle T_{z\bar{z}}\rangle + 2(\partial_z\sigma)\langle T_{z\bar{z}}\rangle = \frac{c}{12\pi}\partial_{\bar{z}}(\partial_z^2\sigma - (\partial_z\sigma)^2) \tag{54}$$

This gives the desired form

$$\langle T_{zz}\rangle = \frac{c}{12\pi}(\partial_z^2\sigma - (\partial_z\sigma)^2)\,. \tag{55}$$

(b) The metric of a Euclidean cylinder $M = \mathbb{R} \times S^1$ (with radius $R = 1$) is given by:

$$ds^2_{\text{cyl}} = d\tau^2 + d\phi^2, \quad \tau \in \mathbb{R},\, \phi \sim \phi + 2\pi.$$

Consider the map from flat space to M:

$$z \mapsto e^{\tau + i\phi}, \quad \bar{z} \mapsto e^{\tau - i\phi}.$$

The metrics relate as:

$$ds^2_{\text{flat}} = dzd\bar{z} = (z\bar{z})(d\tau + id\phi)(d\tau - id\phi) = e^{2\tau} ds^2_{\text{cyl}}.$$

Thus, $(g_{\text{cyl}})_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$ with $\sigma = -\tau$. In $z, \bar{z}$ coordinates, this gives

$$\sigma = -\frac{1}{2}(\ln z + \ln \bar{z}) \tag{56}$$

Thus,

$$\langle T_{zz} \rangle_{\text{cyl}} = \frac{c}{12\pi}\frac{1}{4z^2} \tag{57}$$

This is related to τ, σ coordinates by

$$\langle T_{zz} \rangle_{\text{cyl}} = \frac{1}{4z^2}(\langle T_{\tau\tau} \rangle_{\text{cyl}} - \langle T_{\sigma\sigma} \rangle_{\text{cyl}}) \tag{58}$$

The Weyl anomaly predicts:

$$\langle T^{\mu}_{\mu} \rangle_M = \frac{c}{24\pi} R[M], \tag{59}$$

where $R[M]$ is the Ricci scalar. Since $R[M] = 0$ for the cylinder, it follows:

$$\langle T_{\tau\tau} \rangle_{\text{cyl}} + \langle T_{\phi\phi} \rangle_{\text{cyl}} = 0. \tag{60}$$

Also note that due to the cylinder's isometries (translations in τ and ϕ), $\langle T_{\tau\phi} \rangle_M = 0$. Combining (58) and (60) gives

$$\langle T_{\tau\tau} \rangle = \frac{c}{24\pi} \tag{61}$$

In real time, the metric becomes:

$$ds^2_{\text{cyl}} = -dt^2 + d\phi^2, \quad t = i\tau.$$

The vacuum energy is:

$$H_{\text{vac}} = \int_{\Sigma} \langle T_{tt} \rangle_M = \int_0^{2\pi} d\phi \left(-\frac{c}{24\pi}\right) = -\frac{c}{12}.$$

In fact, in this context it is more standard to use the Weyl anomaly $\langle T^{\mu}_{\mu} \rangle = \frac{c}{12}R$, whereas we used $\langle T^{\mu}_{\mu} \rangle = \frac{c}{24\pi}R$. In the convention where $\langle T^{\mu}_{\mu} \rangle = \frac{c}{12}R$, we would need to multiply $\langle T_{zz} \rangle$ by a factor 2π. This gives

$$H_{\text{vac}} \to -\frac{\pi c}{6} \tag{62}$$

For a cylinder with radius R, this becomes $H = -\frac{\pi c}{6R}$. Thus, c can be measured by the Casimir energy of a critical Hamiltonian on the cylinder.

(c) For a general coordinate transformation $z \mapsto w = f(z, \bar{z})$, $\bar{z} \mapsto \bar{w} = \bar{f}(z, \bar{z})$, the metric components transform as:

$$g_{ww} = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial w}, \quad g_{\bar{w}\bar{w}} = \frac{\partial z}{\partial \bar{w}}\frac{\partial \bar{z}}{\partial \bar{w}}, \quad g_{w\bar{w}} = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial \bar{w}}.$$

To remain Weyl flat, $g_{ww} = g_{\bar{w}\bar{w}} = 0$, which requires:

$$\frac{\partial \bar{z}}{\partial w} = \frac{\partial z}{\partial \bar{w}} = 0, \quad \frac{\partial z}{\partial w}, \frac{\partial \bar{z}}{\partial \bar{w}} \neq 0.$$

This implies the transformations $z = f(w), \bar{z} = \bar{f}(\bar{w})$, leading to:

$$ds^2 = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial \bar{w}} dw d\bar{w}.$$

The Weyl factor is:

$$\sigma = \frac{1}{2}\ln\left(\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}\right).$$

For this scale factor:

$$\partial_z^2\sigma - (\partial_z\sigma)^2 = \frac{1}{2}\frac{w^{(3)}(z)}{w'(z)} - \frac{3}{4}\left(\frac{w''(z)}{w'(z)}\right)^2 = \frac{1}{2}\{w, z\},$$

where $\{w, z\}$ is the Schwarzian derivative. Thus:

$$\langle T \rangle = \frac{c}{24\pi}\{w, z\}.$$

Including this anomalous term recovers the full transformation law.